AD-A137 787 A SOLUTION TECHNIQUE FOR COMPLEX MATRIX GAMES(U)
AERONAUTICAL RESEARCH LABS MELBOURNE (AUSTRALIA)
G J MURRAY APR 83 ARL/SYS NOTE-83

1/1

UNCLASSIFIED F/G 12/1 NL

END
DATE FILMED
3 84
DTIC

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

ARL-SYS-NOTE-83

AR-002-952

(12)

ADA137787

DEPARTMENT OF DEFENCE

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

AERONAUTICAL RESEARCH LABORATORIES

MELBOURNE, VICTORIA

SYSTEMS NOTE 83

A SOLUTION TECHNIQUE FOR COMPLEX MATRIX GAMES

APPROVED

FOR PUBLIC RELEASE

by

G. J. MURRAY

DTIC ELECTE FEB 14 1984 S E D

THE UNITED STATES NATIONAL TECHNICAL INFORMATION SERVICE IS AUTHORISED TO REPRODUCE AND SELL THIS REPORT

© COMMONWEALTH OF AUSTRALIA 1983

COPY No

APRIL 1983

DTIC FILE COPY

84 02 14 074

AR-002-952

DEPARTMENT OF DEFENCE
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION
AERONAUTICAL RESEARCH LABORATORIES

SYSTEMS NOTE 83

A SOLUTION TECHNIQUE FOR COMPLEX MATRIX GAMES

by
G. J. MURRAY

SUMMARY

Recently Mond and Murray introduced the concept of a zero sum two person game in complex space and proved two minimax theorems. In this note a method is provided for the solution of such games. A numerical example is given.

POSTAL ADDRESS: Director, Aeronautical Research Laboratories,
Box 4331, Melbourne, Victoria, 3001, Australia

CONTENTS

Accession For
NTIS GRA&I ☒
DTIC TAB ☐
Unannounced ☐
Justification

By
Distribution/
Availability Codes
Dist | Avail and/or Special
A-1

NOTATION

N is the set $\{1, 2, \ldots, n\}$, the first n positive integers. $x \geqslant y$ means that $x_j \geqslant y_j$ for all $j \in N$ (where x and y are n-dimensional column vectors). $x > y$ means that $x_j > y_j$ for all $j \in N$. $R^{m \times n}(C^{m \times n})$ is the set of real (complex) $m \times n$ matrices. For any $A \in C^{m \times n}$, A^t is the transpose and A^H the conjugate transpose. Re w means the real part of w, e.g., for any $w \in C^n$, Re w denotes the vector whose jth component is Re (w_j). e is a column vector in which every element is 1, the dimension of the vector to be clear from the context. For any nonempty set $S \subset C^m$, $S^* = \{z \in C^m: w \in S \text{ implies that } \mathrm{Re}(z^H w) \geqslant 0\}$ is the *polar* of S, and int. $S^* = \{z \in C^m: 0 \neq w \in S$ implies that $\mathrm{Re}(z^H w) > 0\}$ is the *interior of* S^*.

The vectors common to S and S^* are denoted by $\hat{S} = S \cap S^*$. A nonempty set $S \subset C^m$ is

(i) *convex* if $0 \leqslant \lambda \leqslant 1$ implies that $\lambda S + (1-\lambda)S \subset S$

(ii) a *cone* if $0 \leqslant \lambda$ implies that $\lambda S \subset S$

(iii) a *convex cone* if (i) and (ii) are true

(iv) a *pointed convex cone* if (iii) is true and $S \cap (-S) = \{0\}$

(v) a *polyhedral cone* if $S = BR^k_+$ for some $B \in C^{m \times k}$.

1. INTRODUCTION

The concept of a zero sum two person game in complex space was introduced previously ~~in [1]~~. Complex analogues of payoffs, probability vectors and strategy sets were defined and a minimax theorem established. A simpler proof of a more general minimax theorem was subsequently given in [2]. Both of the above papers provided numerical examples of 2×2 matrix games in complex space and their solutions. Solution of matrix games in complex space has so far been quite difficult, even for 2×2 matrices. In this note a method is derived for transforming a matrix game in complex space into a matrix game in real space having the same value. This latter game can then be solved by the standard techniques involving linear programming. Optimal strategies for the original game in complex space are easily obtained from the optimal strategies computed for the game in real space.

The technique outlined in this note can be used to solve any feasible matrix game in complex space. The only constraint on the size of the game matrix which can be accommodated is that which is imposed by the computer or linear programming package to be used.

2. DEFINITIONS AND PRELIMINARIES

The development of the theory of complex matrix games requires some generalization of real space game concepts.

An n-dimensional *complex probability vector* is a column vector $z \in C^n$ satisfying

(i) $\mathrm{Re}(z_j) \geqslant 0 \; (j = 1, \ldots, n)$

(ii) $\sum_{j=1}^{n} z_j = 1.$

Note that the sum of the imaginary parts of the components of a complex probability vector is zero.

A *strategic cone* $S \subset C^m$ is a pointed polyhedral (closed) convex cone such that $S \subset (R^m_+)^*$ and $S \cap (R^m)^* = \{0\}$. The requirement that $S \subset (R^m_+)^*$ assures that the real parts of all components of any vector $s \in S$ are non-negative while the requirement that $S \cap (R^m)^* = \{0\}$ means that, except for the origin, the complex axes are not in S. The significance of a strategic cone is that any complex probability vector $w \in S$, where S is a strategic cone, will have bounded imaginary components.

Let $S \subset C^m$ be a strategic cone. The set of complex probability vectors $w \in S$, if such vectors exist, will be contained within a strategic cone whose generators $p^j = c^j + id^j$ $(j = 1, \ldots, n)$ satisfy $\sum_{k=1}^{m} c^j_k = 1$ and $\sum_{k=1}^{m} d^j_k = 0$, and are thus complex probability vectors themselves.

The strategic cone generated by $p^j (j = 1, \ldots, n)$ will be denoted by P_S. P_S is the convex conical hull of all complex probability vectors $w \in S$. Since all of these vectors $p^j (j = 1, \ldots, n)$ belong to S, $P_S \subset S$. If S contains R^m_+ then so does P_S. P_S is nonempty if and only if there exists at least one complex probability vector $w \in S$.

Note that $P_{S^*} \neq P_S{}^*$. By definition P_{S^*} does not contain any vectors whose imaginary parts do not sum to zero. $P_S{}^*$ does contain such vectors.

For a complex two person game a *strategy* is defined to be a complex probability vector. Each player selects a strategy from his own *strategy set*. Generally each player makes his choice without any prior knowledge of the other player's choice.

A complex matrix game is completely specified by a payoff matrix A and strategic cones S and T. Let $A \in C^{m \times n}$, $S \subset C^m$, $T \subset C^n$. Player 1 chooses a strategy (complex probability vector) $w \in S$ and player 2 chooses a strategy $z \in T$. Player 1 receives a payoff $\mathrm{Re}(w^H Az)$. Correspondingly player 2 receives a payoff $-\mathrm{Re}(w^H Az)$. The game is *feasible* if there exists at least one strategy $w \in S$ and at least one strategy $z \in T$, i.e. P_S and P_T are nonempty.

A *solution* to complex matrix game $A \in C^{m \times n}$ with strategic cones $S \subset C^m$, $T \subset C^n$ is a pair of complex probability vectors $w_o \in S$ and $z_o \in T$ and a real number v such that

$\mathrm{Re}\, w_o^H Az \geq v$ for all probability vectors $z \in T$, and

$\mathrm{Re}\, w^H Az_o \leq v$ for all probability vectors $w \in S$.

w_o and z_o are *optimal strategies* for players 1 and 2 respectively and v is the *value* of the game.

Note that the existence of a solution (w_o, z_o, v) implies

$$\mathrm{Re}\,(w^H Az_o) \leq \mathrm{Re}\,(w_o{}^H Az_o) \leq \mathrm{Re}\,(w_o{}^H Az)$$

and

$$\max_w . \min_z \mathrm{Re}\,(w^H Az) = \min_z . \max_w \mathrm{Re}\,(w^H Az) = \mathrm{Re}\,(w_o{}^H Az_o) = v.$$

3. THE SOLUTION TECHNIQUE

Consider a matrix game specified by the payoff matrix $A \in C^{m \times n}$ and strategic cones $S \subset C^m$ and $T \subset C^n$. Player 1 chooses a strategy (complex probability vector) $w \in S$ and player 2 chooses a strategy $z \in T$. Player 1 receives a payoff of $\mathrm{Re}\,(w^H Az)$ from player 2.

Let the complex probability vectors $p^j (j = 1, \ldots, n_1)$ be the generators of the convex conical hull of all complex probability vectors $w \in S$. Similarly, let the complex probability vectors $q^k (k = 1, \ldots, n_2)$ be the generators of the convex conical hull of all complex probability vectors $z \in T$. Let a and b be real n_1 and n_2-dimensional probability vectors respectively, and let $P = [p^1 p^2 \ldots p^{n_1}]$ and $Q = [q^1 q^2 \ldots q^{n_2}]$.

Any strategies $w \in S$, $z \in T$ may be written in the form $w = Pa$, $z = Qb$. Note that $w^H = a^t P^H$. Let $M = \mathrm{Re}\,(P^H AQ)$. Then, since a and b are real,

$$\mathrm{Re}\,(w^H Az) = \mathrm{Re}\,(a^t P^H A Q b) = a^t M b.$$

Now, $a^t Mb$ is the payoff obtained for a matrix game in real space with player 1 choosing probability vector a while player 2 chooses probability vector b. This game can be solved using linear programming. Suppose that it has value v and optimal strategies a_o, b_o. Then the original matrix game in complex space has value v and optimal strategies $w_o = Pa_o$ and $z_o = Qb_o$. Also $v = a_o{}^t Mb_o = \mathrm{Re}\,(w_o{}^H Az_o)$. Note that the p^j's and q^k's represent complex analogues of pure strategies.

4. A NUMERICAL EXAMPLE

Consider a matrix game in complex space with the following payoff matrix:

$$\begin{array}{ccc} & & \text{Player 2} \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Player 1} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 1+i \\ 1-i & 3i \\ i & 1-2i \end{bmatrix} \end{array}$$

Let player 1 choose his strategy from strategic cone S and player 2 choose his strategy from strategic cone T. Let S be generated by the set of complex probability vectors

$$\begin{bmatrix} 1+\frac{1}{2}i \\ 0 \\ -\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} 1-\frac{1}{2}i \\ \frac{1}{2}i \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{2}i \\ 1+\frac{1}{2}i \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1-\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -\frac{1}{2}i \\ 1+\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2}i \\ 0 \\ 1-\frac{1}{2}i \end{bmatrix}.$$

Let T be generated by the set of complex probability vectors

$$\begin{bmatrix} 1+\frac{1}{3}i \\ -\frac{1}{3}i \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3}i \\ 1-\frac{1}{3}i \end{bmatrix}, \quad \begin{bmatrix} 1-\frac{1}{3}i \\ \frac{1}{3}i \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{3}i \\ 1+\frac{1}{3}i \end{bmatrix}.$$

For this game P_S (the convex conical hull of S) $= S$ and $P_T = T$.

$$\text{Hence } P = \begin{bmatrix} 1+\frac{1}{2}i & 1-\frac{1}{2}i & -\frac{1}{2}i & 0 & 0 & \frac{1}{2}i \\ 0 & \frac{1}{2}i & 1+\frac{1}{2}i & 1-\frac{1}{2}i & -\frac{1}{2}i & 0 \\ -\frac{1}{2}i & 0 & 0 & \frac{1}{2}i & 1+\frac{1}{2}i & 1-\frac{1}{2}i \end{bmatrix}$$

$$\text{and } Q = \begin{bmatrix} 1+\frac{1}{3}i & \frac{1}{3}i & 1-\frac{1}{3}i & -\frac{1}{3}i \\ -\frac{1}{3}i & 1-\frac{1}{3}i & \frac{1}{3}i & 1+\frac{1}{3}i \end{bmatrix}.$$

$$M = \operatorname{Re}(P^{H}AQ) = \frac{1}{10} \begin{bmatrix} 19 & 29 & 11 & 21 \\ 17 & 22 & 13 & 18 \\ 13 & 18 & -3 & 2 \\ 26 & -19 & 14 & -31 \\ 2 & -23 & 18 & -7 \\ -9 & 21 & -1 & 29 \end{bmatrix}$$

After adding 31 to each element in the matrix, the usual linear programming technique yields optimal strategies for the real matrix game M. These are

$$a_o = \begin{bmatrix} 0 \\ \frac{5}{6} \\ 0 \\ 0 \\ \frac{1}{6} \\ 0 \end{bmatrix} \text{ for player 1 and } b_o = \begin{bmatrix} 0 \\ 0 \\ \frac{5}{6} \\ \frac{1}{6} \end{bmatrix} \text{ for player 2.}$$

Optimal strategies for the original complex matrix game are thus

$$w_o = \begin{bmatrix} \frac{5}{6}-\frac{5}{12}i \\ \frac{1}{3}i \\ \frac{1}{6}+\frac{1}{12}i \end{bmatrix} \text{ for player 1 and } z_o = \begin{bmatrix} \frac{5}{6}-\frac{1}{3}i \\ \frac{1}{6}+\frac{1}{3}i \end{bmatrix} \quad \text{for player 2.}$$

The common value of the two games is $\frac{83}{60}$.

5. REMARKS

The complex space conversion of a game to a linear program has been demonstrated in [3]. There is in fact an equivalence between matrix games and linear programs in complex space [4]. Hence the above solution technique for matrix games in complex space suggests that it may be possible to convert a linear programming problem in complex space into a real linear program. The conversion is readily accomplished [5], thus providing a method for solving linear programs in complex space.

REFERENCES

[1] Mond, B., and Murray, G. J. (1982). Game Theory in Complex Space. Opsearch, **19**, 1–11.

[2] Mond, B., and Murray. G. J (1983). A Minimax Theorem for Matrix Games in Complex Space. Opsearch, *20*, 25–34.

[3] Murray, G. J. (1978). Game Theory in Complex Space, M.A. Thesis, Latrobe University, Melbourne, Australia.

[4] Mond, B., and Murray, G. J. Matrix Games and Linear Programs in Complex Space. Draft paper (to be published).

[5] Mcnd, B., and Murray, G. J. On Solving Complex Programs. Draft paper (to be published).

DISTRIBUTION

AUSTRALIA

DEPARTMENT OF DEFENCE

Central Office

Chief Defence Scientist
Deputy Chief Defence Scientist
Superintendent, Science and Technology Programmes
Controller, Projects and Analytical Studies
} 1 copy

Defence Science Representative (U.K.) (Doc. Data sheet only)
Counsellor, Defence Science (U.S.A.) (Doc. Data sheet only)
Defence Central Library
Document Exchange Centre, D.I.S.B. (17 copies)
Joint Intelligence Organisation
Librarian H Block, Victoria Barracks, Melbourne
Director General—Army Development (NSO) (4 copies)

Aeronautical Research Laboratories

Director
Library
Superintendent—Systems
Divisional File—Systems
Author: G. J. Murray
S. Pelczynski

Materials Research Laboratories

Director/Library

Defence Research Centre

Library

RAN Research Laboratory

Library

Navy Office

Navy Scientific Adviser

Army Office

Army Scientific Adviser

Air Force Office

Air Force Scientific Adviser

Central Studies Establishment

Information Centre

DEPARTMENT OF DEFENCE SUPPORT

Government Aircraft Factories
Manager
Library

DEPARTMENT OF AVIATION
Library

STATUTORY AND STATE AUTHORITIES AND INDUSTRY
SEC of Vic., Herman Research Laboratory, Library
B.H.P., Melbourne Research Laboratories
Major Furnace and Combustion Engineers Pty. Ltd., Manager

UNIVERSITIES AND COLLEGES

Adelaide	Barr Smith Library
Latrobe	Library
	Professor B. Mond, Pure Mathematics Department
Melbourne	Engineering Library
	Dr B. Craven, Mathematics Department
Monash	Hargrave Library
Newcastle	Library
Sydney	Engineering Library
N.S.W.	Physical Sciences Library
Queensland	Library
Western Australia	Library
R.M.I.T.	Library

CANADA
NRC
Aeronautical & Mechanical Engineering Library

Universities and Colleges

Toronto	Institute for Aerospace Studies

FRANCE
ONERA, Library

INDIA
Defence Ministry, Aero Development Establishment, Library
Hindustan Aeronautics Ltd., Library
National Aeronautical Laboratory, Information Centre

ISRAEL
Technion-Israel Institute of Technology
Professor J. Singer

JAPAN
Institute of Space and Astronautical Science, Library

Universities

Kagawa University	Professor H. Ishikawa

NETHERLANDS

National Aerospace Laboratory (NLR), Library

NEW ZEALAND

Universities

Canterbury	Library

SWEDEN

Aeronautical Research Institute, Library
Swedish National Defence Research Institute (FOA)

UNITED KINGDOM

Royal Aircraft Establishment
Bedford, Library
National Physical Laboratory, Library
National Engineering Laboratory, Library
British Library, Lending Division
Fulmer Research Institute Ltd., Research Director
British Hovercraft Corporation Ltd., Library

Universities and Colleges

Manchester	Professor, Applied Mathematics
Nottingham	Science Library
Southampton	Library
Strathclyde	Library
Cranfield Inst. of Technology	Library

UNITED STATES OF AMERICA

NASA Scientific and Technical Information Facility
The John Crerar Library
Allis Chalmers Corporation, Library
Lockheed-California Company
Lockheed Missiles and Space Company
Lockheed Georgia

Universities and Colleges

Massachusetts Inst. of Technology	M.I.T. Libraries

SPARES (50 copies)

TOTAL (136 copies)

Department of Defence

DOCUMENT CONTROL DATA

1. a. AR No.	1. b. Establishment No.	2. Document Date	3. Task No.
AR-002-952	ARL-SYS-NOTE-83	April, 1983	—

4. Title	5. Security	6. No. Pages
A SOLUTION TECHNIQUE FOR COMPLEX MATRIX GAMES	a. document: Unclassified	5
	b. title: U. c. abstract: U.	7. No. Refs: 5

8. Author(s)	9. Downgrading Instructions
G. J. Murray	—

10. Corporate Author and Address	11. Authority (~~as appropriate~~) a. Sponsor b. Security c. Downgrading d. Approval
Aeronautical Research Laboratories, P.O. Box 4331, Melbourne, Vic. 3001.	—

12. Secondary Distribution (*of this document*

Approved for public release

Overseas enquirers outside stated limitations should be referred through ASDIS, Defence Information Services, Branch, Department of Defence, Campbell Park, CANBERRA, ACT 2601.

13. a. This document may be ANNOUNCED in catalogues and awareness services available to . . .

No limitations

13. b. Citation for other purposes (*i.e. casual announcement*) may be ~~(select)~~ unrestricted ~~(or) as for 13 a.~~

14. Descriptors		15. COSATI Group
Game theory	Linear programming	1201
Matrix games	Complex space	

16. Abstract

Recently Mond and Murray introduced the concept of a zero sum two person game in complex space and proved two minimax theorems. In this note a method is provided for the solution of such games. A numerical example is given.

This page is to be used to record information which is required by the Establishment for its own use but which will not be added to the DISTIS data base unless specifically requested.

16. Abstract (Contd)		
17. Imprint Aeronautical Research Laboratories, Melbourne		
18. Document Series and Number Systems Note 83	19. Cost Code 74 0012	20. Type of Report and Period Covered —
21. Computer Programs Used		
22. Establishment File Ref(s) —		

END

DATE
FILMED

3 84

DTIC